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Wilson Loops, Winding modes and Domain Walls in Finite Temperature QCD

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We discuss the effective action for Polyakov-Wilson loops winding around compact Euclidean time, which serve as order parameters for the finite temperature deconfinement transition in $SU(N)$ Yang-Mills gauge theory. We then apply our results to the study of the high temperature continuation of the confining phase, and to the analysis of certain Z_N domain walls that have been argued to play a role in cosmology. We argue that the free energy of these walls is much larger than previously thought.

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1. Introduction

There are numerous reasons to study $SU(N)$ gauge theory at finite temperature. In particular, detailed understanding of the $SU(3)$ theory should be useful in describing the physics of QCD in the early universe, and properties of the quark-gluon plasma [1] (the high temperature phase) are important for relativistic heavy ion collisions. More formally, one is interested in understanding the deconfinement transition at large N and study its relation to the ideas of Hagedorn [2] and to string theory [3]. This is especially interesting since the perturbative gauge field description, suitable at high temperature and the string one, valid at low temperature (in the confining phase), should be complimentary to each other.

As we will briefly review later, the free energy $e^{-\beta F} = \mathrm{Tr} e^{-\beta H}$ and other physical quantities are given at finite temperature by a path integral over gauge fields living on the Euclidean manifold $R^3 \times S^1$ with Euclidean time x^0 identified with $x^0 + \beta$; gauge fields are periodic: $A_\mu(x^0 + \beta, \mathbf{x}) = A_\mu(x^0, \mathbf{x})$, while quarks (if present) are anti-periodic. There is a nice correspondence between the phase structure of $SU(N)$ gauge fields and Z_N spin systems [4]. The high temperature (deconfined) phase in gauge theory corresponds to the ordered (low temperature) phase of the spin system and vice versa. The scalar gauge invariant order parameters which capture the dynamics of gauge fields are time-like Polyakov – Wilson loops:

$$W_n(\mathbf{x}) = \frac{1}{N} \mathrm{Tr} e^{i \int_0^{n\beta} A_0(x^0, \mathbf{x}) dx^0} \tag{1.1}$$

(with the trace in the fundamental representation of $SU(N)$). Only $N - 1$ of the W_n are independent. Their correlation functions are of utmost importance; e.g. the quark-antiquark free energy $F_{q,\bar{q}}$ defined by:

$$e^{-\beta F_{q,\bar{q}}(\mathbf{x},\mathbf{y})} = \langle W_1(\mathbf{x}) W_{-1}(\mathbf{y}) \rangle \tag{1.2}$$

measures the free energy of a system with a static quark at $\mathbf{x}$ and an antiquark at $\mathbf{y}$. Higher W_n are related to higher representations of $SU(N)$. In the confining phase $\langle W_n \rangle = 0$ and $F_{q,\bar{q}}(\mathbf{x}) \sim |\mathbf{x}|$ as $|\mathbf{x}| \to \infty$, while in the deconfined phase $\langle W_n \rangle \neq 0$ and $F_{q,\bar{q}}(\mathbf{x}) \sim$ constant as $|\mathbf{x}| \to \infty$.

To focus on the dynamics of the W_n one may integrate out the other degrees of freedom and study an effective action of the general form:

$$\begin{aligned} S_{\text{eff}} =& N^2\bigg[\int d^3\mathbf{x}d^3\mathbf{y}\sum_n W_n(\mathbf{x})W_{-n}(\mathbf{y})G^{(2)}_n(\mathbf{x}-\mathbf{y}) \\ &+\int d^3\mathbf{x_1}d^3\mathbf{x_2}d^3\mathbf{x_3}\sum_{n_1,n_2} W_{n_1}(\mathbf{x_1})W_{n_2}(\mathbf{x_2})W_{-n_1-n_2}(\mathbf{x_3})G^{(3)}_{n_1,n_2}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})+\cdots\bigg]. \end{aligned} \tag{1.3}$$

The kernels $G^{(2)}, G^{(3)}, \ldots$ summarize the dynamics, and are the object of this paper. Eq. (1.3) is the analog of a Landau-Ginzburg description of the spin system, and can be used [5] in the usual way to study the behavior of $F_{q,\bar{q}}$ (1.2) and other observables. At large N the structure is similar to string theory. Terms of high order in W_n are suppressed by powers of $g_{\text{string}} = 1/N$. In particular, it is very interesting to compare the inverse propagator for Wilson loops $G^{(2)}$ to the corresponding quantity in conventional string theories; knowledge of $G^{(2)}$ is sufficient to deduce the value of the large N deconfinement (Hagedorn) transition (β_H) and other physical properties, like the string tension as a function of temperature.

The immediate motivation for our work is related to three recent ideas:

1) In [6], J. Polchinski proposed to study properties of the confining phase (essentially properties of $G^{(2)}$ (1.3)) of four dimensional Yang-Mills theory using perturbation theory, valid at high temperature, by "analytically continuing" the confining phase to that regime. These arguments were since generalized to other cases [7]; there have also been attempts to match the resulting behavior of $G^{(2)}$ to particular string theories (strings with Dirichlet boundaries [8], and rigid strings [9]). As pointed out by Polchinski [6], it is surprising to obtain properties of an essentially non-perturbative object (the confining phase) from perturbation theory. Our study of (1.3) will allow us to clarify this issue somewhat.

2) It has been proposed [10], that at high temperature, when one knows that the minima of S_{eff} are at $\langle W_n\rangle = e^{\frac{2\pi inj}{N}}, j = 0, 1, \ldots, N-1$, the system may possess domain walls separating regions in space with different $\langle W_n\rangle$; these domain walls may be of cosmological interest. The interface tension (energy per unit area), α, of such domain walls has been argued to be perturbatively calculable and to go like $\alpha \sim \frac{T^3}{g(T)}$ [11] (with T the temperature and g the running gauge coupling). If valid, these arguments could suggest interesting new phenomena that may have occurred in the early universe [12]. In addition, these domain walls would correspond to non – perturbative effects of order $\exp(-1/g)$ in (finite temperature) gauge theory, which would be of more general theoretical interest [13]. However, various objections to this scenario have been raised [14], [15]. In particular, it has been

claimed [16] that infrared (IR) divergences may lead to subtleties in the arguments of [11], although the leading behavior $\alpha \sim \frac{T^3}{g(T)}$ was argued to be safe. We will reconsider some of these issues below.
3) In [7], [17] it has been argued that two dimensional QCD coupled to adjoint "quarks" may be a good toy model of the relation between large N gauge theory and strings, exhibiting a non – trivial spectrum of "Regge trajectories" and a large N deconfinement transition. It is of interest to develop techniques that would allow calculations of properties of this Hagedorn transition. The results of this investigation will be reported separately [18].

In this paper we are going to discuss the structure of the effective action $S_{\rm eff}$, presenting techniques to evaluate $G^{(k)}$ perturbatively in the gauge coupling. The plan of the paper is as follows. Section 2 is devoted to a brief review of finite temperature gauge theory, presented mainly to set the notations and specify the necessary calculations. In section 3 we describe the calculation of $G^{(2)}$ to one-loop order and outline the calculations of higher order kernels ($G^{(3)}, \cdots$). The perturbative analysis is seen to be infrared finite in a certain region. We discuss separately the contributions to $G^{(2)}$ of gluons and other kinds of adjoint matter.

In sections 4, 5 we discuss the lessons learned from the one-loop calculations of section 3 for the two problems mentioned above, of the high temperature limit of the confining phase, and the interface tension of domain walls. We show that infrared divergences do not alter the analysis of [6], [7] (contrary to recent claims), but unfortunately one does not appear to be able to learn much about properties of the confining phase from this analysis. On the other hand, for the domain wall problem infrared issues are found to play a crucial role, and in fact alter the qualitative behavior of the domain wall energy per unit area α [11], from $\alpha \sim \frac{1}{g}$ to $\alpha \sim \frac{1}{g^2}$. Section 6 contains a summary of our conclusions and necessary future work.

2. General Formalism

It is customary [5], [19] to calculate quantities like

$$Z(\beta) = {\rm Tr} e^{-\beta H} \tag{2.1}$$

in the $A_0 = 0$ gauge. Ignoring matter fields for simplicity, we have for the QCD Lagrangian $\mathcal{L} = \frac{1}{g^2}{\rm Tr}F^2$ the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\int d^3\mathbf{x}[g^2(\mathbf{E}^a)^2 + \frac{1}{g^2}(\mathbf{B}^a)^2] \tag{2.2}$$

where $\mathbf{E}^a, \mathbf{B}^a$ are the color electric and magnetic fields respectively; $\mathbf{E}^a, \mathbf{A}^a$ are canonically conjugate. The physical Hilbert space is spanned by $|\mathbf{A}(x)\rangle$ satisfying Gauss' Law constraint

$$\mathbf{D} \cdot \mathbf{E}|\text{phys}\rangle = 0. \tag{2.3}$$

This constraint may be enforced via a Lagrange multiplier:

$$Z = \int \mathcal{D}\mathbf{A}(x)\langle \mathbf{A}|e^{-\beta\mathcal{H}}P|\mathbf{A}\rangle \tag{2.4}$$

where $P = \int \mathcal{D}\Gamma(x) e^{i\int d^3\mathbf{x}\mathrm{Tr}D\Gamma(x)\cdot\mathbf{E}(x)}$ projects on solutions of Gauss' law (2.3). Standard Feynman path integral manipulations then lead to the expression:

$$Z = \int [\mathcal{D}A_\mu(x)] e^{-\frac{1}{g^2}\int_0^\beta dx^0 \int d^3\mathbf{x}\mathrm{Tr}F_{\mu\nu}^2} \tag{2.5}$$

where Γ has been renamed A_0 and the gauge fields have the periodicity properties:

$$A_\mu(x^0 + \beta, \mathbf{x}) = A_\mu(x^0, \mathbf{x}). \tag{2.6}$$

One can also study more sophisticated questions like what is the free energy in the presence of sources. For that one needs to generalize the Gauss' law constraint (2.3) to $\mathbf{D} \cdot \mathbf{E} = \rho$. Repeating the previous discussion one finds [5], [19] that the free energy with static quarks at positions $\mathbf{x_1}, \ldots, \mathbf{x_n}$ and antiquarks at $\mathbf{y_1}, \ldots, \mathbf{y_n}$ is given in terms of Euclidean time-like Wilson loops (1.1) by:

$$e^{-\beta F(\mathbf{x_1},\ldots,\mathbf{x_n},\mathbf{y_1},\ldots,\mathbf{y_n})} = \langle \prod_{i=1}^{n} W_1(\mathbf{x_i})W_{-1}(\mathbf{y_i})\rangle \tag{2.7}$$

with the average performed in the measure (2.5). We see that the dynamics of the W_n encoded in (1.3) summarizes many important physical properties of the theory.

The theory (2.5) is invariant under gauge transformations $A_\mu \to U^{-1}A_\mu U + U^{-1}\partial_\mu U$ such that $U(x^0 + \beta, \mathbf{x}) = zU(x^0, \mathbf{x})$. The prefactor z is constrained by the periodicity of A_μ (2.6) to lie in the center of the gauge group. For $SU(N)$ the center is simply Z_N and $z = e^{\frac{2\pi i n}{N}}, n = 0, 1, \ldots, N-1$. Local gauge invariant observables are invariant under these aperiodic transformations, while W_n (1.1) transforms as:

$$W_n \to z^n W_n. \tag{2.8}$$

The effective action (1.3) must exhibit the global Z_N symmetry (2.8), but this symmetry can be broken spontaneously. Indeed, in the confining ("disordered") phase where this ("center") Z_N symmetry is unbroken, $\langle W_n \rangle = 0$, while at high temperature $\langle W_n \rangle \neq 0$ and Z_N is spontaneously broken just as in the ordered phase of the spin models [4].

To calculate quantities like (2.7) in the continuum, one may proceed as follows. First fix the gauge; (2.5), (2.6) does not allow going to $A_0 = 0$ gauge. A convenient gauge choice is

$$\bar{A}_0^{ab}(x^0, \mathbf{x}) = \frac{2\pi}{\beta} \theta_a(\mathbf{x}) \delta^{ab}. \tag{2.9}$$

Thus $\bar{A}_0$ is diagonal and independent of x_0 and $\sum_{a=1}^{N} \theta_a = 0$ (mod 1). Then integrate out the spatial gauge fields A_i in (2.5), and whatever (adjoint) matter fields are present, and find an effective action of the general form (1.3) written in terms of $\theta_a(\mathbf{x})$ or (1.1):

$$W_n(\mathbf{x}) = \frac{1}{N} \sum_{a=1}^{N} e^{2\pi i n \theta_a(\mathbf{x})}. \tag{2.10}$$

The description in terms of $\theta_a(\mathbf{x})$ is natural in the deconfined phase (at high temperatures) where, as we will see, the effective action is minimized for certain fixed θ_a, so that $\langle W_n \rangle \neq 0$. Below the deconfinement transition the θ_a are randomly distributed and $\langle W_n \rangle = 0$, thus it is preferable to describe the dynamics in terms of W_n. We will use perturbation theory in the gauge coupling g, which in principle should be reliable at high temperatures (up to possible IR divergences) due to the running of the gauge coupling; nevertheless, we will mostly write the effective action in terms of W_n (2.10), in the spirit of [6]; the Wilson loops seem to be the suitable variables for discussing the Hagedorn transition [18], and may teach us something about high temperature string theory. To study the deconfinement transition and/or the confining phase one must use non-perturbative tools [18].

3. The One-Loop Effective Action

We will be mostly interested in two general classes of gauge systems:

1) Two dimensional Yang-Mills coupled to adjoint (bosonic or fermionic) matter [7], [17]. The Lagrangian for scalar matter is

$$\mathcal{L} = (D_\mu \phi)^2 + m^2 \phi^2 + \frac{1}{g^2} F^2 \tag{3.1}$$

where

$$D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi]; \quad F = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1]. \tag{3.2}$$

We choose the gauge (2.9); the Faddeev-Popov determinant is $\det(\partial_0 + i\bar{A}_0)$. Adding to this the effect of integrating out A_1, ϕ to one-loop order we find

$$e^{-S_{\rm eff}^{1-\rm loop}(\bar{A}_0)} = \left[\det[(\partial_0 + i\bar{A}_0)^2]\right]^{-\frac{1}{2}} (\det(-\bar{D}^2 + m^2))^{-\frac{1}{2}} \det(\partial_0 + i\bar{A}_0). \tag{3.3}$$

with the three determinants on the r.h.s. arising from integration over A_1, ϕ and the ghosts, respectively. The ghost contribution exactly cancels that of A_1 and we are left with

$$S_{\rm eff}^{1-\rm loop}(\bar{A}_0) = \frac{1}{2} \log\det(-\bar{D}^2 + m^2). \tag{3.4}$$

For constant $\bar{A}_0$ one can use the results of [19], [20] to evaluate $S_{\rm eff}(\bar{A}_0)$. We will discuss the determinant for an $\mathbf{x}$ dependent $\bar{A}_0$ (2.9).

For fermionic (Majorana) adjoint matter one finds

$$S_{\rm eff}^{1-\rm loop}(\bar{A}_0) = -\frac{1}{4} \log\det(-D^2 \mathbf{1} + F_{\mu\nu} J^{\mu\nu} + m^2 \mathbf{1}) \tag{3.5}$$

where $J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are the Lorentz generators and $D_\mu = \partial_\mu - iA_\mu$.

2) D-dimensional pure Yang-Mills (with D>2). Thus we start with

$$\mathcal{L} = \frac{1}{g^2} F_{\mu\nu}^2. \tag{3.6}$$

Here, it is convenient to use a background field gauge and Feynman gauge for the quantum fields. The one-loop effective action in this gauge is

$$S_{\rm eff}^{1-\rm loop}(\bar{A}_0) = \frac{1}{2} \log\det(-D^2 \mathbf{1} + F_{\mu\nu} J^{\mu\nu}) - \log\det(-D^2) \tag{3.7}$$

with the Lorentz generators $(J_{\mu\nu})^{\rho\sigma} = i(\eta_\mu^\rho \eta_\nu^\sigma - \eta_\nu^\rho \eta_\mu^\sigma)$. The second term on the right hand side of (3.7) is due to the ghosts, whose contribution is minus that of a (complex) massless scalar in the adjoint representation.

The one-loop effective actions (3.4), (3.5), (3.7) should be evaluated for $A_\mu = \bar{A}_0 \delta_{\mu 0}$ (2.9) and added to the classical action $S_{\rm eff}^{\rm clas} = \frac{\beta}{g^2} \int d^{D-1}\mathbf{x} (\nabla \bar{A}_0)^2$.

Most of the analysis will be done for the case of scalar adjoint matter, which (at one-loop) has most of the qualitative features of the other two cases. The properties special to the other cases will be mentioned later [1].

[1] The following two subsections are rather technical. Readers interested only in the results may proceed directly to eq. (3.34).

3.1. The one-loop effective action for scalar adjoint matter - general considerations

It is useful to use a first quantized representation to calculate the determinant (3.4) (see e.g [21] for a recent review). The effective action is given by a path integral over worldline trajectories $x_\mu(t), 0 \leq t \leq T$ that wind n times around the compact x_0 direction:

$$\mathbf{x}(t+T) = \mathbf{x}(t); \quad x_0(t+T) = x_0(t) + n\beta. \tag{3.8}$$

One has also to sum over the length of the world line $T \in \mathbf{R}^+$ and winding number $n \in \mathbf{Z}$. Thus the one-loop effective action (3.4) can be written as:

$$S_{\text{eff}}^{1-\text{loop}}[\bar{A}_0] = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \text{Tr} \int_0^\infty \frac{dT}{T}\mathcal{N}\int [\mathcal{D}x(t)] \exp\left[-\int_0^T dt(\frac{1}{4}\dot{x}^2 - iA_0[x(t)]\dot{x}_0) - m^2 T\right] \tag{3.9}$$

with the trace taken in the adjoint representation and $\mathcal{N}$ a normalization factor. This form of expressing the determinant (3.4) is related to the standard Feynman diagram representation by Poisson resummation. It is convenient to exhibit the n dependence explicitly by taking $x_0(t) \to x_0(t) + \frac{n\beta t}{T}$ such that $x_0(t)$ is now periodic and the effective action becomes:

$$\begin{aligned} S_{\text{eff}}^{1-\text{loop}}[\bar{A}_0] =& -\frac{1}{2}\sum_{n=-\infty}^{\infty} \text{Tr} \int_0^\infty \frac{dT}{T}\mathcal{N}\int [\mathcal{D}x(t)] \\ & \exp\left[-\int_0^T dt\left(\frac{1}{4}\dot{x}^2 + \frac{n^2\beta^2}{4T^2} - iA_0[x(t)]\left(\dot{x}_0(t) + \frac{n\beta}{T}\right)\right) - m^2 T\right]. \end{aligned} \tag{3.10}$$

The path integral over $x(t)$ for a general A_0 (2.9) defines a complicated quantum mechanics problem (which is closely related to the amplitude for pair production in a general external electric field), but fortunately we do not need the full solution, at least when trying to set up the expansion (1.3) in powers of W_n. As a warm up exercise, consider the simple case of constant A_0 (or θ_a (2.9)). Due to the periodicity of x_0 we have:

$$\int_0^T dt A_0\left(\dot{x}_0(t) + \frac{n\beta}{T}\right) = n\beta A_0,$$

and (replacing $SU(N)$ by $U(N)$, which is unimportant at large N):

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{1-\text{loop}}(\theta_a) =& -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{a,b=1}^{N}\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{dT}{T^{1+\frac{d}{2}}} e^{-\frac{n^2\beta^2}{4T^2} - m^2 T} e^{2\pi i n(\theta_a - \theta_b)} \\ =& -N^2 \sum_n c_n W_n W_{-n} \end{aligned} \tag{3.11}$$

where $c_n = \left(\frac{m}{2\pi n\beta}\right)^{\frac{d}{2}} K_{\frac{d}{2}}(nm\beta)$ and K_n is a modified Bessel function. For $m = 0, d = 4$ this gives the result of [19] (for constant A_0 there is no difference between the scalar (3.4) and the gauge (3.7) determinants), while for $d = 2$ it agrees with [7]. We also notice that the winding of the adjoint scalar around compact time n in (3.10) is identical to the winding of the Wilson line n in (1.3) (using (2.10)). This is a general feature of the one-loop determinants (as we will see below) which will make things easier later. Note also that for *constant* A_0 (or $W_n(1.1)$) the effective action (1.3) is exactly quadratic to one-loop order, i.e. $G^{(3)} = G^{(4)} = \ldots = 0$. We will see soon that this is not the case for non-zero momentum and for higher orders in the gauge coupling.

For arbitrary $A_0(x)$ one may proceed as follows. Write

$$x(t) = x_{cl} + x_q(t) \tag{3.12}$$

where x_{cl} is constant, and x_q is the fluctuating quantum mechanical variable in (3.10), which is Gaussian with:

$$\langle x_q^\mu(t_1) x_q^\nu(t_2)\rangle = -G(t_1 - t_2) g^{\mu\nu}; \quad G(t+T) = G(t) = |t| - t^2/T. \tag{3.13}$$

Now substituting (3.12) in (3.10) and expanding (for simplicity we take A_0 to depend on only one of the $d-1$ spatial directions; no generality is lost because of rotational invariance) we get:

$$A_0(x_{cl} + x_q) = A_0(x_{cl}) + A_0'(x_{cl}) x_q + \frac{1}{2} A_0''(x_{cl}) x_q^2 + \cdots + \frac{1}{n!} A_0^{(n)}(x_{cl}) x_q^n + \cdots. \tag{3.14}$$

We will write this compactly as $A_0[x(t)] = A_0[x] + \tilde{A}_0[x(t)]$ where we use x to denote x_{cl} and $x(t)$ to denote $x_{cl} + x_q$.

Now expand (3.10) in powers of $\tilde{A}_0$ (or θ_a) and use (3.13) and (3.14) to average over quantum fluctuations of the trajectory. We find that $S_{\text{eff}}^{1-\text{loop}}[A_0]$ is naturally expressed in terms of bilinears in quantities of the form ($\theta_a^{(n)}$ denotes the n'th derivative of θ_a):

$$V^{(n)}_{n_1,\ldots,n_k} = \frac{1}{N} \sum_{a=1}^{N} \theta_a^{(n_1)} \cdots \theta_a^{(n_k)} e^{2\pi i n \theta_a(x_{cl})}; \qquad n_i \geq 1, \tag{3.15}$$

i.e.

$$\mathcal{L}_{\text{eff}}^{1-\text{loop}}[V] = N^2 \sum_n a_{\{n_i\}\{m_j\}} V^{(n)}_{\{n_i\}} \bar{V}^{(n)}_{\{m_j\}}. \tag{3.16}$$

This structure follows immediately from properties of the trace in the adjoint representation (for $U(N)$). A Wilson loop in the adjoint representation of $U(N)$ can be written as a product of a Wilson loop in the fundamental N and one in the anti-fundamental $\bar{N}$ representation, i.e.

$$\begin{aligned}
\mathcal{L}_{\rm eff}^{1-\rm loop} =& -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum\int_0^\infty \frac{dT}{T^{\frac{d}{2}+1}} e^{-\frac{n^2\beta^2}{4T}-m^2T}\left\langle \mathrm{Tr}_{\rm adj}\exp\left[i\int_0^T dtA_0\cdot\left(\dot{x}_0+\frac{n\beta}{T}\right)\right]\right\rangle \\
=& -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum\int_o^\infty \frac{dT}{T^{\frac{d}{2}+1}} e^{-\frac{n^2\beta^2}{4T}-m^2T} \\
& \left\langle \mathrm{Tr}_{\rm N}\exp\left[i\int_0^T dtA_0\cdot\left(\dot{x}_0+\frac{n\beta}{T}\right)\right]\mathrm{Tr}_{\rm N}\exp\left[i\int_0^T dtA_0\cdot\left(\dot{x}_0+\frac{n\beta}{T}\right)\right]^\dagger\right\rangle .
\end{aligned} \tag{3.17}$$

Now for the gauge choice (2.9), each Wilson loop in the fundamental representation can be written in the form (3.15) by first expanding the gauge field as in (3.14) and then expanding the exponent in powers of $\tilde{\theta}$, i.e.

$$\begin{aligned}
&\mathrm{Tr}_N\exp\left[i\int_0^T dtA_0\cdot\left(\dot{x}_0(t)+\frac{n\beta}{T}\right)\right] = \sum_{a=1}^N e^{2\pi in\theta_a(x)}e^{\frac{2\pi i}{\beta}\int_0^T dt\tilde{\theta}_a(x(t))\left(\dot{x}_0(t)+\frac{n\beta}{T}\right)} \\
=&\sum_{a=1}^N e^{2\pi in\theta_a(x)}\Bigg[1+\frac{2\pi i}{\beta}\int_0^T dt\tilde{\theta}_a(x(t))\left(\dot{x}_0(t)+\frac{n\beta}{T}\right) \\
&-\frac{4\pi^2}{\beta^2}\int_0^T dt_1\int_0^{t_1} dt_2\tilde{\theta}_a(x(t_1))\tilde{\theta}_a(x(t_2))\left(\dot{x}_0(t_1)+\frac{n\beta}{T}\right)\left(\dot{x}_0(t_2)+\frac{n\beta}{T}\right)+\cdots\Bigg] \\
=&\sum_{a=1}^N e^{2\pi in\theta_a(x)}\Bigg[1+\frac{2\pi i}{\beta}\int_0^T dt\left(\theta'_a(x)x_q(t)\left(\dot{x}_0(t)+\frac{n\beta}{T}\right)+\cdots\right) \\
&-\frac{4\pi^2}{\beta^2}\int_0^T dt_1\int_0^{t_1} dt_2\left(\theta'_a(x)\theta'_a(x)x_q(t_1)x_q(t_2)\left(\dot{x}_0(t_1)+\frac{n\beta}{T}\right)\left(\dot{x}_0(t_2)+\frac{n\beta}{T}\right)+\cdots\right) \\
&+\cdots\Bigg] \\
=&N\Bigg[W^{(n)}+\frac{2\pi i}{\beta}V_1^{(n)}\int_0^T dt\left(x_q(t)\left(\dot{x}_0(t)+\frac{n\beta}{T}\right)\right) \\
&-\frac{4\pi^2}{\beta^2}V_{1,1}^{(n)}\int_0^T dt_1\int_0^{t_1} dt_2\left(x_q(t_1)x_q(t_2)\left(\dot{x}_0(t_1)+\frac{n\beta}{T}\right)\left(\dot{x}_0(t_2)+\frac{n\beta}{T}\right)\right)+\cdots\Bigg].
\end{aligned} \tag{3.18}$$

This expansion can be thought of as an expansion around slowly varying A_0. One obtains a similar term for the anti-fundamental Wilson loop and together they give a one-loop effective action which is written in terms of bilinears of $V_{\{n_i\}}$ (3.16).

It is not clear at first sight how to rewrite the complicated functions $V_{\{n_i\}}$ (3.15) in terms of the Wilson loops $W_n(x)$ (2.10), which is a necessary step for constructing the expansion (1.3), although it is clear that up to global issues the W_n's exhaust the degrees of freedom of the θ_a out of which the $V_{\{n_i\}}$ are constructed. It turns out that one can construct an expansion of $V_{\{n_i\}}$ in a power series in W_n. To leading order in W_n the results are simple; in fact one can show that for $k \geq 2$ in (3.15),

$$V^{(n)}_{n_1,\cdots,n_k} = O(W^k); \tag{3.19}$$

whereas for k=1 we have:

$$V^{(n)}_{n_1} = \frac{1}{2\pi i n}\left(\frac{\partial}{\partial x}\right)^{n_1} W_n(x) + O(W^2). \tag{3.20}$$

The terms of order W^l $l \geq 2$ above are of course calculable as well. We will not derive (3.19), (3.20) here. To illustrate the flavor of the arguments, we consider the simplest non-trivial case of $V_{\{n_i\}}$ with $\sum_{i=1}^k n_i = 2$. There are in this case only two functions

$$\begin{aligned} \tilde{V}^{(n)}_2 \equiv & Z_n = \frac{1}{N}\sum_{a=1}^N 2\pi i n \theta''_a e^{2\pi i n\theta_a} \\ V^{(n)}_{1,1} \equiv & L_n = \frac{1}{N}\sum_{a=1}^N (\theta'_a)^2 e^{2\pi i n\theta_a}. \end{aligned} \tag{3.21}$$

Clearly (2.10)

$$W''_n = Z_n - (2\pi n)^2 L_n. \tag{3.22}$$

To illustrate (3.20) we have to show that $L_n = O(W^2)$.

To establish that, consider

$$\begin{aligned} 0 = X_n = & \frac{1}{N^2}\sum_{a,b=1}^N \sum_{k=-\infty}^{\infty} (\theta'_a - \theta'_b)^2 e^{2\pi i n\theta_a - 2\pi i k(\theta_a - \theta_b)} \\ & = 2(L_n + L_0 W_n) + 2\sum_{k\neq 0,n}[L_k W_{n-k} + \frac{1}{4\pi^2 k(n-k)} W'_k W'_{n-k}]. \end{aligned} \tag{3.23}$$

All the L_n are expected to be of the same order in W; we see from (3.23) that this order must be $O(W^2)$ and that:

$$L_n = -\sum_{k\neq 0,n}\frac{1}{4\pi^2 k(n-k)} W'_k W'_{n-k} + O(W^3). \tag{3.24}$$

Similar considerations allow one to prove (3.19), (3.20) for all n_i.

The importance of the observations (3.19), (3.20) is that to select terms in (3.16) which are bilinear in W (and thus contribute to $G^{(2)}$ (1.3)), we need to keep only terms with $k=1$ (3.20) [2]. In terms of the expansion in $\tilde{\theta}$ (see (3.18)) this implies that one need only take terms up to first order in $\tilde{\theta}$ from each fundamental trace (or up to second order in $\tilde{\theta}$ if the trace is done in the adjoint representation). Higher order (in W_n) contributions to the effective action (1.3) can be derived systematically by using (3.23), (3.24) and generalizations to expand the $V_{\{n_i\}}$ in a power series in W_n. The fact that the winding n contribution to the one-loop effective action (3.10) is directly related to $G_n^{(2)}$ in (1.3) together with the above observations allow one to calculate $G_n^{(2)}$ to this order. We now turn to this calculation.

3.2. Calculation of the inverse propagator for Wilson loops $G_n^{(2)}$

Starting from (3.17), we have to expand in $\tilde{\theta}$, average over the fluctuating x_q and rewrite the results in terms of bilinears in W_n. Using the results of the previous subsection it is easy to see that:

$$S_{\text{eff}}^{1-\text{loop}}[\theta] = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=-\infty}^{\infty}\sum_{a,b=1}^{N}\int_0^\infty \frac{dT}{T^{1+\frac{d}{2}}}e^{-\frac{n^2\beta^2}{4T}-m^2T+2\pi i n\theta_{ab}(x)}$$
$$\left\langle 1+\frac{8\pi^2}{\beta^2}\int_0^T dt_1\int_0^{t_1} dt_2\left(\tilde{\theta}_a[x(t_1)]\tilde{\theta}_b[x(t_2)]\left(\dot{x}_0(t_1)+\frac{n\beta}{T}\right)\left(\dot{x}_0(t_2)+\frac{n\beta}{T}\right)\right)\right\rangle \tag{3.25}$$

where we defined $\theta_{ab}\equiv\theta_a-\theta_b$ and neglected $O(W^3)$ terms.

Now we need to expand $\tilde{\theta}[x(t_i)]$ (3.14) and average over the x_q. Only those terms with equal numbers of derivatives in the expansion of the $\tilde{\theta}_a\tilde{\theta}_b$ contribute after averaging over quantum fluctuations:

$$S_{\text{eff}}^{1-\text{loop}}[\theta] = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=-\infty}^{\infty}\sum_{a,b=1}^{N}\int_0^\infty \frac{dT}{T^{1+\frac{d}{2}}}e^{-\frac{n^2\beta^2}{4T}-m^2T+2\pi i n\theta_{ab}(x)}$$
$$\left(1+\frac{8\pi^2}{\beta^2}\int_0^T dt_1\int_0^{t_1} dt_2\left(\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\theta_a^{(l)}(x)\theta_b^{(l)}(x)G(t_1-t_2)^l(\ddot{G}(t_1-t_2)+\frac{n^2\beta^2}{T^2})\right)\right) \tag{3.26}$$

where both derivative in $\ddot{G}$ are with respects to t_1. Note that the $n=0$ contribution to the sum vanishes, since $\sum\theta_a(x)=0$ (mod 1).

[2] Except for the $n=0$ contribution which will be discussed separately later.

We can shift the variables of integration to make the integral independent of t_2 due to the periodicity of the propagators G. Doing the remaining integral over t_1 gives a factor of $T^l \frac{l!l!}{(2l+1)!}$:

$$
\begin{aligned}
S_{\text{eff}}^{1-\text{loop}}[\theta] = &-\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{a,b=1}^{N}\sum_{n\neq 0}\int_0^\infty \frac{dT}{T^{1+\frac{d}{2}}}e^{-\frac{n^2\beta^2}{4T}-m^2T+2\pi i n\theta_{ab}(x)} \\
&\left(1+\frac{8\pi^2T^2}{\beta^2}\sum_{l=1}^{\infty}\frac{(-T)^l l!}{(2l+1)!}\theta_a^{(l)}(x)\theta_b^{(l)}(x)(\left(\frac{n\beta}{T}\right)^2-\frac{2}{T})\right).
\end{aligned}
\tag{3.27}
$$

Next we change variables to $u=\frac{2m^2}{\alpha}T$ where $\alpha=|n|\beta m$. The last equation can easily be written in terms of the W_n by using (3.20). For convenience we write the answer in momentum space (an integral over momenta is implied):

$$
\begin{aligned}
S_{\text{eff}}^{1-\text{loop}}[W] = &-\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n\neq 0}\left(\frac{2m^2}{\alpha}\right)^{\frac{d}{2}}\int_0^\infty \frac{du}{u^{1+\frac{d}{2}}}e^{-\frac{\alpha}{2}(u+\frac{1}{u})}W_n(k)W_{-n}(-k) \\
&\left(1-2\sum_{l=1}^{\infty}\frac{(-1)^l l!}{(2l+1)!}\left(\frac{k^2\alpha u}{2m^2}\right)^l(\frac{u}{\alpha}-1)\right).
\end{aligned}
\tag{3.28}
$$

The integral over u can be expressed in terms of modified Bessel functions using the defining equation

$$
K_\nu(x)=\frac{1}{2}\int_0^\infty du u^{\nu-1}e^{-\frac{x}{2}(u+\frac{1}{u})}.
$$

From this we obtain

$$
\begin{aligned}
S^{1-\text{loop}}{}_{\text{eff}}[W] = &-\sum_{n\neq 0}\left(\frac{m^2}{2\pi\alpha}\right)^{\frac{d}{2}}W_n(k)W_{-n}(-k) \\
&\left(K_{\frac{d}{2}}(\alpha)+2\sum_{l=1}^{\infty}\frac{(-1)^l l!}{(2l+1)!}\left(\frac{\alpha k^2}{2m^2}\right)^l(K_{l-\frac{d}{2}}(\alpha)-\frac{1}{\alpha}K_{l-\frac{d}{2}+1}(\alpha))\right).
\end{aligned}
\tag{3.29}
$$

As mentioned above, to complete the calculation we have to evaluate the zero winding, $n=0$, contribution. For constant θ_a the zero winding sector gives rise to an unimportant temperature independent constant, which is usually dropped. In general, however, the $n=0$ contribution is non – trivial. Indeed, to quadratic order in W_n one finds:

$$
S_{\text{eff}}^{1-\text{loop}}[\theta]_{n=0} = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{a,b=1}^{N}\int_0^\infty \frac{dT}{T^{1+\frac{d}{2}}}e^{-m^2T}\left(1+\frac{16\pi^2T}{\beta^2}\sum_{l=1}^{\infty}\frac{(-T)^l l!}{(2l+1)!}\theta_a^{(l)}(x)\theta_a^{(l)}(x)\right)
\tag{3.30}
$$

where half of the contribution is due to the fundamental and half to the anti-fundamental representations.

We need to write this in terms of non-zero winding Wilson loops W_n, which can be achieved by utilizing the derivation of (3.24) and generalizations. In momentum space, one finds:

$$S_{\text{eff}}^{1-\text{loop}}[W]_{n=0} = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n\neq 0} \frac{2m^2}{\alpha^2} W_n(k)W_{-n}(-k) \int_0^\infty \frac{dT}{T^{\frac{d}{2}}} e^{-m^2 T} \sum_{l=1}^{\infty} \frac{(-Tk^2)^l l!}{(2l+1)!} \tag{3.31}$$

where we have dropped a temperature independent constant, and as usual a momentum integral is implied.

Finally, doing the T integral gives

$$S_{\text{eff}}^{1-\text{loop}}[W]_{n=0} = \left(\frac{m^2}{4\pi}\right)^{\frac{d}{2}} \sum_{n\neq 0} \frac{2}{\alpha^2} W_n(k)W_{-n}(-k) \left(\sum_{l=1}^{\infty} \frac{(-1)^l l! \Gamma(l-\frac{d}{2}+1)}{(2l+1)!} \left(\frac{k^2}{m^2}\right)^l\right). \tag{3.32}$$

We can now combine (3.29) with (3.32) and find the one-loop contribution to the quadratic action for Wilson loops:

$$\begin{aligned} S_{\text{eff}}^{1-\text{loop}}[W] = &- \sum_{n\neq 0} \left(\frac{m^2}{4\pi}\right)^{\frac{d}{2}} W_{-n}(-k)W_n(k) \Bigg(\left(\frac{2}{\alpha}\right)^{\frac{d}{2}} K_{\frac{d}{2}}(\alpha) \\ &+ \sum_{l=1}^{\infty} \frac{(-1)^l l!}{(2l+1)!} \left(\frac{k^2}{m^2}\right)^l \left[\left(\frac{\alpha}{2}\right)^{l-\frac{d}{2}-1} (\alpha K_{l-\frac{d}{2}}(\alpha) - K_{l-\frac{d}{2}+1}(\alpha)) + \frac{2}{\alpha^2}\Gamma(l-\frac{d}{2}+1)\right]\Bigg). \end{aligned} \tag{3.33}$$

Adding this to the term in the effective action coming from the classical action we get $G_n^{(2)}$ to one-loop order:

$$\begin{aligned} G_n^{(2)}(k) = &\frac{1}{(4\pi)^{\frac{d}{2}}} \frac{m^2}{\alpha^2} \frac{k^2}{2g^2 N} - \left(\frac{m^2}{4\pi}\right)^{\frac{d}{2}} \Bigg(\left(\frac{2}{\alpha}\right)^{\frac{d}{2}} K_{\frac{d}{2}}(\alpha) \\ &+ \sum_{l=1}^{\infty} \frac{(-1)^l l!}{(2l+1)!} \left(\frac{k^2}{m^2}\right)^l \left[\left(\frac{\alpha}{2}\right)^{l-\frac{d}{2}-1} (\alpha K_{l-\frac{d}{2}}(\alpha) - K_{l-\frac{d}{2}+1}(\alpha)) + \frac{2}{\alpha^2}\Gamma(l-\frac{d}{2}+1)\right]\Bigg). \end{aligned} \tag{3.34}$$

Recall that $\alpha = m\beta|n|$. Eqns. (3.33), (3.34) are the main result of this section. We now turn to the study of some of their properties.

3.3. Properties of $G_n^{(2)}(k)$

Despite appearances, the limit $m \to 0$ of (3.34) is actually regular. To study this limit it is convenient to rewrite the sum over Bessel functions in terms of a compact integral by reversing the order of the sum and integral in (3.27). This gives

$$
\begin{aligned}
G_n^{(2)}(k) = & \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{m^2}{\alpha^2} \frac{k^2}{2g^2N} - \left(\frac{m^2}{2\pi\alpha^2}\right)^{\frac{d}{2}} \Bigg(\alpha^{\frac{d}{2}} K_{\frac{d}{2}}(\alpha) \\
& + \int_0^\infty \frac{du}{u^{1+\frac{d}{2}}} e^{-\frac{1}{2}(\alpha^2 u + \frac{1}{u})} f(\frac{\alpha^2 k^2 u}{2m^2}) - \int_0^\infty \frac{du}{u^{\frac{d}{2}}} (e^{-\frac{1}{2}(\alpha^2 u + \frac{1}{u})} - e^{-\frac{1}{2}\alpha^2 u}) f(\frac{\alpha^2 k^2 u}{2m^2}) \Bigg).
\end{aligned}
\tag{3.35}
$$

where for $x > 0$

$$f(x) = -\frac{1}{\sqrt{x}} e^{-\frac{x}{4}} \int_0^x dt \frac{\sqrt{t}}{2} e^{\frac{t}{4}} \tag{3.36}$$

and for $x < 0$

$$f(x) = \frac{1}{\sqrt{|x|}} e^{\frac{|x|}{4}} \int_0^{|x|} dt \frac{\sqrt{t}}{2} e^{-\frac{t}{4}}. \tag{3.37}$$

To study the convergence properties of the integrals in (3.35) we have to examine the asymptotic behavior of $f(x)$. First note that in both cases $f(x) \sim -x$ as $x \to 0$. Thus, for $d < 4$ the integrals in (3.35) are convergent in the UV (for small u). The second integral exhibits a standard logarithmic UV divergence which corresponds to coupling constant renormalization in $d = 4$. For $x > 0$, it is easy to see that $f(x)$ goes to a constant (-2) as $x \to \infty$. Hence for $k^2 > 0$ the integral representation for $G^{(2)}$ (3.35) is convergent even for m=0 (and fixed β), for which the exponential IR suppression of the integrals (provided by the mass) is absent. This is to be contrasted with the infrared divergent results one finds if one expands f in a power series in $\frac{\alpha^2}{m^2}k^2u$ (derivative expansion) and integrates term by term.

In the case $x < 0$ as $|x| \to \infty$ the integral in (3.37) converges to $2\sqrt{\pi}$. This implies that $f(x)$ grows as $\frac{e^{\frac{|x|}{4}}}{\sqrt{|x|}}$ as $x \to -\infty$. Putting this asymptotic form for $f(x)$ into (3.35) for $k^2 < 0$ and large u gives

$$G^{(2)} \sim \left(\frac{m^2}{\alpha^2}\right)^{\frac{d}{2}} \sqrt{\frac{m^2}{\alpha^2|k|^2}} \int^\infty \frac{du}{u^{\frac{d+3}{2}}} \exp\left[-\frac{\alpha^2 u}{2}\left(1 - \frac{|k|^2}{4m^2}\right)\right] \tag{3.38}$$

For $k^2 < -4m^2$ this integral diverges. By rescaling the integration variable we see that in this range $G^{(2)}$ develops a branch cut: $G^{(2)} \propto (4m^2 - |k|^2)^{(d+1)/2}$. We will discuss this cut below and will see that it has some important consequences.

As mentioned above, we find a regular expression for $G^{(2)}$ as $m \to 0$. This is easiest to see from (3.35), in the limit $\alpha \to 0$, $\alpha/m = \beta|n|$ fixed. We find:

$$\begin{aligned} G_n^{(2)}(k) =& \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{n^2\beta^2} \frac{k^2}{2g^2N} - \left(\frac{1}{4\pi n^2\beta^2}\right)^{\frac{d}{2}} \Bigg(2^{d-1} \\ &+ \int_0^\infty \frac{du}{u^{1+\frac{d}{2}}} e^{-\frac{1}{4u}} f(n^2\beta^2k^2u) - 2\int_0^\infty \frac{du}{u^{\frac{d}{2}}} (e^{-\frac{1}{4u}} - 1) f(n^2\beta^2k^2u)\Bigg). \end{aligned} \tag{3.39}$$

By using the asymptotic behavior of (3.36), (3.37) we see that $G_n^{(2)}(k^2 > 0)$ is regular. The branch cut mentioned above starts now at $k^2 = 0$.

Before going on to apply these results to physical problems, in the next subsections we briefly comment on similar calculations for gauge fields and fermionic matter.

3.4. Fermionic matter

For fermionic matter we start from (3.5) and remembering that the fermions have anti-periodic boundary conditions, i.e. $p_0 = \frac{\pi}{\beta}(2n+1)$, we derive the analog of (3.10). Since the γ matrices are anti-commuting operators, it is natural in this first quantized approach to introduce world line fermions to represent them. The easiest way to do this is to implement a supersymmetric generalization of (3.8), introducing superpartners $\psi_\mu(t)$ for the $x_\mu(t)$. For more detail see [21].

For adjoint fermionic matter in two dimensions we obtain

$$\begin{aligned} &S_{\text{eff}}^{1-\text{loop}}[\bar{A}_0] = \frac{1}{4}\sum_{n=-\infty}^{\infty}(-1)^n \text{Tr}\int_0^\infty \frac{dT}{T}\mathcal{N}\int[\mathcal{D}x(t)][\mathcal{D}\psi(t)] \\ &\exp\left[-\int_0^T dt\left(\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\cdot\dot{\psi} + 2i\psi^0F_{01}\psi^1 + \frac{n^2\beta^2}{4T^2} - iA_0[x(t)]\left(\dot{x}_0(t) + \frac{n\beta}{T}\right)\right) - m^2T\right] \end{aligned} \tag{3.40}$$

where $F_{01} = \partial_1 A_0$.

We need to expand to second order in A_0 and then average over the quantum fluctuations of the path, as well as the ψ fields. Note that the ψ fields satisfy

$$\begin{aligned} \langle\psi^\mu(t_1)\psi^\nu(t_2)\rangle =& -G_F(t_1 - t_2)g^{\mu\nu} \\ -G_F(t+T, t') = G_F(t, t') =& \text{sgn}(t - t'). \end{aligned} \tag{3.41}$$

After some algebra we obtain

$$S_{\rm eff}^{1-\rm loop}[A] = \frac{1}{4}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=-\infty}^{\infty}(-1)^n {\rm Tr}\int_0^\infty \frac{dT}{T^{1+\frac{d}{2}}} e^{-\frac{n^2\beta^2}{4T}-m^2T+in\beta A(x)}$$
$$\Big(1-\int_0^T dt_1\int_0^{t_1} dt_2\Big(\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}(A^{(l)}(x))^2[G(t_1-t_2)]^{l-1}(-lG_F(t_1-t_2)^2$$
$$+G(t_1-t_2)(\ddot{G}(t_1-t_2)+\frac{n^2\beta^2}{T^2}))\Big)\Big). \tag{3.42}$$

The rest of the analysis follows straightforwardly from the bosonic case. The term introduced by the world sheet fermions does not effect the position of the cut in the effective action.

3.5. Gauge fields

We write here only the one-loop effective action obtained from integrating out gauge fields in four dimensions. The interested reader may consult the work of Strassler [21] for a thorough discussion on how to handle vector fields in this approach. The effective action is given by

$$S_{\rm eff}^{1-\rm loop}[A] = -\frac{1}{4}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=-\infty}^{\infty}{\rm Tr}\int_0^\infty \frac{dT}{T^3} e^{-\frac{n^2\beta^2}{4T}+in\beta A(x)}$$
$$\Big(1-\int_0^T dt_1\int_0^{t_1} dt_2\Big(\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}(A^{(l)}(x))^2[G(t_1-t_2)]^{l-1} \tag{3.43}$$
$$(-4l+G(t_1-t_2)(\ddot{G}(t_1-t_2)+\frac{n^2\beta^2}{T^2}))\Big)\Big).$$

The rest of the analysis is similar to the scalar case with the cut starting at $k^2=0$ as expected.

4. Consequences for the high temperature continuation of the confining phase

In reference [6], J. Polchinski has proposed to use the form of $G^{(2)}$ (1.3) to study the confining phase of QCD. The idea, inspired by string theory, is to view the perturbative calculation of $G^{(2)}$, $G^{(3)}, \ldots$ which is in principle only valid at high temperature, in the deconfined phase, as an analytic continuation of the confined phase to high temperatures.

In string theory such analytic continuations are routine; the analogs of the Wilson lines W_n, which are winding modes around compact time are governed by an action:

$$\mathcal{L}_{str} = \frac{1}{g_{st}^2} W_n(\mathbf{k}) W_{-n}(-\mathbf{k})[\mathbf{k}^2 + M_n^2(\beta)] + O(W^3); \qquad M_n^2(\beta) = \beta^2 n^2 - C \tag{4.1}$$

with C a positive constant. There is no difficulty in formally extending this formula to $\beta < \beta_H = \sqrt{C}$.

It has been pointed out [22] that calculations such as those of [6], [7] in which $G^{(2)}$ is calculated only at zero momentum may suffer from IR divergences, since to find $S_{\rm eff}$ in that case, massless particles are sometimes integrated out, and the effective action is in principle non-local already at $\mathbf{k}^2 \approx 0$. Of course, in four dimensional gauge theory the spatial gauge fields that are integrated out are not really massless but rather develop a magnetic mass [19] $m \approx g^2(\beta)/\beta$ (which is unfortunately incalculable in perturbation theory), and in adjoint QCD_2 one may turn on by hand a mass for the matter fields (whose role is precisely to mimic the above magnetic mass). One would however like to understand whether the results of [6], [7] are reliable in the massless limit, and more importantly whether one can indeed learn about the confining phase from such perturbative calculations.

The results of section 3 help to resolve both issues. Regarding the IR divergences, we see that they are harmless, even for massless adjoint matter (gluons) in $2d$ ($4d$). Consider for example the case of massless, bosonic adjoint QCD_2. The inverse propagator $G^{(2)}$ vanishes when (3.39):

$$1 - \frac{k^2}{4g^2N} + \frac{1}{2}\int_0^\infty \frac{du}{u^2} e^{-\frac{1}{4u}} f(k^2\beta^2n^2u) - \int_0^\infty \frac{du}{u}(e^{-\frac{1}{4u}} - 1) f(k^2\beta^2n^2u) = 0 \tag{4.2}$$

(see section 3 for definition of f and derivation). In [6], [7] the last two terms on the r.h.s. of (4.2) were neglected, because they are formally small at high temperature (of higher order in β^2g^2) when the sum of the first two vanishes, i.e. when $k^2 = 4g^2N$. We saw in section 3 that the IR singularity due to integrating out a massless field results in a cut to the *left* of $k^2 = 0$. This cut is not dangerous at $4g^2N = k^2 > 0$. The full inverse propagator (4.2) vanishes at $k^2 = 4g^2N[1 + O(g^2\beta^2)]$ (up to logarithmic corrections), approaching the result of [6], [7] as the temperature goes to infinity. To put it differently, integrating out the massless constituents does not introduce infrared subtleties since the Wilson loops W_n are tachyonic at high temperature. Thus the calculations of [6] and [7] are valid, contrary to the recent claims [22].

Unfortunately, our result (3.33) seems, at least naively, to invalidate the basic idea of using perturbation theory for $G^{(l)}$ to study the confining phase of QCD. Our calculations give a (one loop) glimpse of the analytic structure of the effective action (1.3) in momentum space. Indeed, one of the most important qualitative differences between the confining and deconfined phases of QCD is the analyticity properties of the Green's functions, such as $G^{(2)}(k)$. In the confining, low temperature phase, $G^{(2)}$ is expected to be an analytic function of $\mathbf{k^2}$ to leading order in $1/N$ (compare to (4.1)), since any singularities would have to be interpreted in terms of interactions of the W_n among themselves and with other singlet bound states and would be down by powers of the string coupling $1/N$. In the high temperature phase, one expects the structure to change drastically. The W_n are no longer the natural degrees of freedom, and in particular there are non-singlet operators (quarks, gluons) that may couple to W_n. Thus one expects $G^{(2)}$ to contain branch cuts corresponding to pair production of such non-singlet degrees of freedom and to have a complicated analytic structure typical of the deconfined phase already in *leading order in* $1/N$.

To determine whether the perturbative calculations of [6], [7] correspond to an analytic continuation from the confining phase or to properties of the deconfined phase one has to study the analytic structure of $G^{(l)}$ and in particular look for branch cut singularities signaling the propagation of constituents. The cuts we found in section 3 are precisely of this kind; they seem to correspond to coupling of W_n to two constituents (quarks, gluons). Thus, the natural conclusion from our analysis is that the perturbative calculations performed in [6], [7] and section 3 give the effective action for A_0 in the *deconfined phase*, written in terms of the variables W_n, and not, as one would hope, the analytic continuation of the confining phase[3]. The latter would have a very different analytic structure than the former, and than what we find (section 3). One can not easily infer any properties of the confining phase (other than it being unstable at high temperature) from these calculations. In particular the number of degrees of freedom of the string does not seem to be easily extractable. There does not seem to be a simple way to calculate in the confining phase without actually following $G^{(2)}$ up to the phase transition at β_H, where all the non-singlet singularities should disappear. This requires a non – perturbative analysis, which may be feasible in certain toy models [18].

[3] In principle one has to investigate the analytic structure of higher order (in g) corrections to (1.3), but it is clear physically that the qualitative picture presented here should persist to all orders.

5. Consequences for the structure of domain walls

In the deconfined phase it is natural to express the action in terms of the θ variables (2.9) as opposed to Wilson loops. For slowly varying $\theta_a(x)$, the Lagrangian (1.3) takes the form (to one loop order):

$$\mathcal{L} = \frac{4\pi^2}{\beta^2 g^2} \sum_{a=1}^{N} (\nabla \theta_a)^2 + V_{\rm eff}(\theta) \tag{5.1}$$

where the kinetic term comes from the classical Lagrangian (3.6) and the one loop potential $V_{\rm eff}$ is (3.11), which in four dimensions is (for $0 \leq \theta_{ab} \leq 1$):

$$\begin{aligned} V_{\rm eff}(\theta) =& -\frac{1}{\pi^2 \beta^4} \sum_{a,b=1}^{N} \sum_{n=-\infty}^{\infty} \frac{1}{n^4} e^{2\pi i n \theta_{ab}} \\ =& \frac{2\pi^2}{3\beta^4} \sum_{a,b=1}^{N} B_4(\theta_{ab}) \end{aligned} \tag{5.2}$$

where $\theta_{ab} = \theta_a - \theta_b$ and $B_4(x)$ is the Bernoulli polynomial

$$B_4(x) = x^4 - 2x^3 + x^2 - 1/30$$

The Lagrangian (5.1) is invariant under the Z_N symmetry (2.8) $\theta_a \rightarrow \theta_a + k/N$. This symmetry implies that there are in fact N different minima of the potential $V_{\rm eff}$, corresponding to all $\theta_a = k/N$ for $k = 0, 1, \ldots, N-1$. In terms of Wilson loops, these minima correspond to $W_n = e^{\frac{2\pi i n k}{N}}$.

By analogy with spin systems [4], it is natural to study domain walls separating regions in space corresponding to different vacua of $V_{\rm eff}$[4]. In particular, one can attempt to calculate the interface energy α, defined as the free energy per unit area of the wall.

At high temperature, when $g(\beta)$ is small, one may hope to use semiclassical techniques to study these walls [11]. To calculate the free energy of a domain wall between regions in space corresponding to $W_n = 1$ and $W_n = \exp(2\pi i n/N)$, say, we have to find a solution of the effective action (5.1) which behaves as $\theta_a(z \rightarrow -\infty) = 0$, $\theta_a(z \rightarrow \infty) = 1/N$ (for all a). It can be shown [11] that the minimal action solution is obtained by choosing a particular path in θ space, parametrized by[5]:

$$\begin{aligned} \theta_a =& q(z)/N, \qquad a = 1, \ldots, N-1, \\ \theta_N =& -\frac{N-1}{N} q(z). \end{aligned} \tag{5.3}$$

[4] There is some debate in the literature regarding the existence and physical significance of such walls [10]– [12], [14]– [16]. We will assume that they exist and study some of their features.

[5] Note that in this section N is not assumed to be large.

In this parameterization the Wilson loop takes the form

$$W_n = e^{\frac{2\pi i n q}{N}}(1 + \frac{1}{N}(e^{2\pi i n q} - 1)).$$

Thus $q = 0$ corresponds to $W_n = 1$ while $q = 1$ corresponds to $W_n = e^{\frac{2\pi i n}{N}}$. The action (5.1) for $q(z)$ (5.3) is (L_t^2 is the transverse area of the domain wall):

$$S_{\text{eff}} = \frac{L_t^2 4(N-1)\pi^2}{\beta} \int dz \left(\frac{1}{g^2 N}(q')^2 + \frac{1}{3\beta^2}[q]^2(1-[q])^2 \right). \tag{5.4}$$

$[q] \equiv q$ mod 1. The solution with the right boundary conditions is

$$q(z) = \frac{\exp(\sqrt{\frac{N}{3}} gz/\beta)}{1 + \exp(\sqrt{\frac{N}{3}} gz/\beta)} \tag{5.5}$$

Plugging it back into (5.4) we find that the interface tension is

$$\frac{S_{\text{eff}}}{\beta L_t^2} = \alpha = \frac{4(N-1)\pi^2}{3\sqrt{3N}} \frac{1}{\beta^3 g}. \tag{5.6}$$

It is clear from (5.5) that the scale of variation of the solution $q(z)$ is $\sim 1/g$. Hence, each derivative comes with a power of g. Formally, this implies that higher derivative terms in the effective action (as well as higher loop contributions) modify the solution (5.5) and α (5.6) only slightly, as the effective coupling at high temperature $g(\beta)$ is small. One might worry [16] that infrared effects due to integrating out massless gluons may spoil the formal power counting. To examine this issue one has to look at higher derivative terms in the effective action for $q(z)$.

Our results are in general insufficient for this task, since we have not calculated S_{eff} for arbitrary $q(z)$. However, to study the above infrared issues it is enough to consider the behavior of the domain wall profile $q(z)$ as $z \to -\infty$ (say), since then q (5.5) is small, $q(z) \simeq \exp(\sqrt{\frac{N}{3}} \frac{gz}{\beta})$, and one can use our results from section 3 to write the effective action to order q^2 (but to all orders in the derivative expansion (3.14))[6]:

$$\begin{aligned} S_{\text{eff}}(q) =& \frac{L_t^2 4(N-1)\pi^2}{\beta} \int dz q(z) \Bigg(-\frac{\partial_z^2}{g^2 N} + \frac{1}{3\beta^2} \\ &+ \frac{1}{2\pi^2\beta^2} \sum_{n\neq 0} \frac{1}{n^2} \Bigg(\frac{1}{4}\int_0^\infty \frac{du}{u^3} e^{-\frac{1}{4u}} f(-\partial_z^2 \beta^2 n^2 u) - \frac{1}{2}\int_0^\infty \frac{du}{u^2}(e^{-\frac{1}{4u}} - 1) f(-\partial_z^2\beta^2 n^2 u) \\ &- n^2\beta^2\partial_z^2 \int_0^\infty \frac{du}{u}(e^{-\frac{1}{4u}} - 1)(f(-\partial_z^2\beta^2 n^2 u) + 1) \Bigg) \Bigg) q(z) + O(q^3, \cdots). \end{aligned} \tag{5.7}$$

[6] The modifications to eq. (5.7) as compared to eq. (3.39) are due to the difference between the scalar (3.4) and gauge (3.7) determinants (see section 3), and should not matter for the qualitative remarks that follow.

We see that the situation is different from that of section 4. Plugging in the asymptotic form of (5.5) in (5.7) we find that $k^2 = -\partial_z^2 = -(N/3)g^2/\beta^2 < 0$, so the terms on the second and third lines of (5.7) are not small corrections to the "leading behavior" obtained from the first two terms on the r.h.s. Unlike the W_n in section 4, the mass squared of q is positive (and equal to the square of the electric mass). Therefore, the branch cut discussed in section 3 significantly alters the behavior coming from (5.1). It seems at first sight that (5.7) completely eliminates the possibility of the existence of a domain wall, since $q(z) = \exp(az)$ is not a solution for any real a. However, the correct interpretation is different.

One important effect due to higher order contributions in g that we have neglected so far is the generation of a "magnetic mass" for the spatial components of the gauge field (static magnetic fields are screened). This magnetic mass, which is of order $m_{\mathrm{mag}} \simeq g^2/\beta$ is not perturbatively calculable [23], [19] but its presence alters the picture following from (5.7). As we saw in section 3, it shifts the branch cut from $k^2 = 0$ to $k^2 = -4m_{\mathrm{mag}}^2 \simeq -4g^4/\beta^2$. The scale of the solution (5.5) of [11] is on the other hand the "electric mass" $m_{\mathrm{el}} \simeq g/\beta$. Since g is small in high temperature QCD, $m_{\mathrm{mag}} \ll m_{\mathrm{el}}$, so that even with the modification of (5.7) due to the magnetic mass, the solution (5.5) is not infrared stable. However, as is clear from (3.35) (with $m = m_{\mathrm{mag}}$, and an appropriate generalization for the gauge case), there is now a solution $\tilde{q}(z)$ which behaves asymptotically as $\tilde{q}(z) \simeq \exp(az)$ with $a \simeq m_{\mathrm{mag}}$.

The main question now is whether the scale of variation of the domain wall solution remains m_{mag} for moderate q as well. In [16] it has been argued that the answer to this question is negative since the infrared scale is determined by the mass of the space – like gluons $\mathbf{A}_{ab}$ which for generic θ_a is $\sqrt{m_{\mathrm{mag}}^2 + (\frac{\theta_{ab}}{\beta})^2}$. In that case, the form of the domain wall solution $q(z)$ (5.5) is only significantly modified at the tails $|z| \to \infty$, and the leading behavior of the interface tension (5.6) is not effected [16]. However, we believe that the scale of variation of the solution $q(z)$ is of order m_{mag} throughout the wall. The point is that there are many physical states whose masses are of order m_{mag} for generic θ (or q). Examples include the space – like gluons $\mathbf{A}_{ab}$ with $a = b$, and gauge invariant combinations like $Tr\mathbf{F}^2$. These can be pair produced at higher orders in the loop expansion and lead, as explained above, to a domain wall profile which varies on the scale $m \approx m_{\mathrm{mag}}$ for all q.

Thus, we conclude that infrared effects change the scale of the domain wall solution of [11] from m_{el} (5.1), (5.5) to $m \approx m_{\mathrm{mag}}$. Accordingly, the interface tension α, (5.6) changes from $\alpha \simeq \frac{1}{g\beta^3}$ to $\alpha \simeq \frac{1}{g^2\beta^3}$. This can be easiest seen by computing the contribution of the

potential term to the effective action (5.4), using the fact that $q \simeq q(mz)$ and $m \approx m_{\text{mag}}$. Higher terms in the effective action give subleading contributions to the interface tension. The free energy of the domain wall seems to be much larger than previously believed, and is in accord with the expected non – perturbative behavior of asymptotically free gauge theory.

To actually prove the above assertion one would need to derive the effective action for finite q and verify that it admits a finite action domain wall solution with the above described asymptotics. Due to the non – perturbative nature of m_{mag} and other problems this seems difficult at present.

6. Conclusion

The main purpose of this paper was to study the properties of the effective action for Wilson loops winding around compact time in different finite temperature gauge theories. The main results obtained are:

1) We have calculated the quadratic term in the effective action for the Wilson loop W_n to one loop order in the gauge coupling constant, and have outlined the calculation of higher order (in W) terms. We found that the inverse propagator of Wilson loops $G^{(2)}$ (1.3), (3.34) contained a branch cut which was interpreted as due to pair production of constituents in the external W_n field. The improved understanding of the dynamics of Wilson loops was then used to reconsider two recent proposals to apply high temperature perturbation theory to different physical problems.

2) We have discussed the idea [6] (see also [7]) that one can use perturbative techniques in QCD to deduce properties of the confining phase, "analytically continued" to high temperature. We argued that the analytic structure of the quadratic term in the effective action for Wilson loops that we found does not support such an interpretation of the perturbative calculations. The inverse propagator in the confining phase is not expected to exhibit any singularities in momentum space, to leading order in $1/N$, while such singularities do appear in the perturbative results.

3) We found that certain domain walls between different vacua with broken Z_N symmetry that were extensively studied in recent literature [10] – [12], [14] – [16] are modified significantly due to infrared effects. In particular, we argued that the free energy of such domain walls behaves like $1/g^2$ as opposed to $1/g$ as previously believed.

There are many natural extensions of this work. One would like to extend the results obtained here to higher orders in the gauge coupling. In order to study the large N deconfinement (Hagedorn) transition one needs, as we saw, to calculate the Green's functions in (1.3) to all orders in g. This may be feasible in toy models of lower dimensional Yang Mills theory [18], where one may use extensions of our techniques to calculate the Hagedorn temperature and perhaps to explicitly see the change in the analytic structure of the action (1.3) between the deconfined and confining phases. The physics of the branch cuts in the Wilson loop propagator found above, and in particular their relation to ones that appear in the true high temperature vacuum (at different k^2 in general), also needs to be understood much better.

It would be interesting to understand what are the implications of the larger free energy of domain walls found here for the cosmological scenarios described in the literature [10], [12]. Also, a better understanding of the dynamics of Wilson loops may suggest ways to study the disintegration of fundamental strings into their purported constituents suggested in [24]. In particular, it would be interesting to obtain a similar picture to that found here for unified strings above the Hagedorn temperature.

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